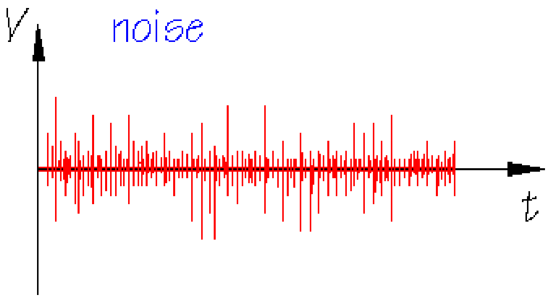
EE 508 Lecture 37

Digital Filters

Review from last lecture

Dynamic Range



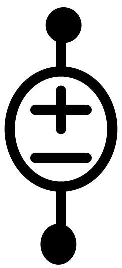
Numerous definitions for DR include some "qualitative" terms in the definition making it difficult to identify a universally accepted definition of the DR though the concept is useful

SNDR is a metric that is rigorously defined that captures some of the DR properties

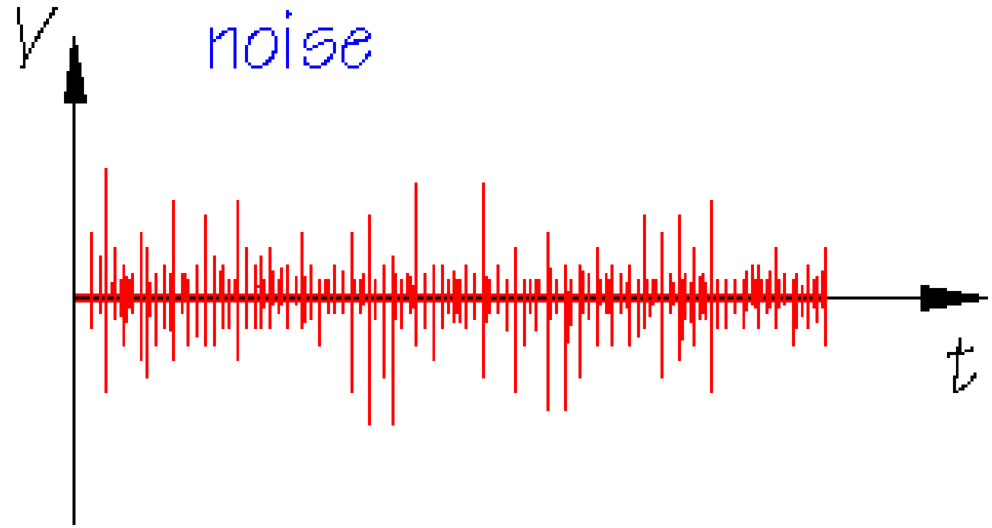
Though the concept of DR is often not discussed rigorously and though there are various definitions of DR, Dynamic Range should be the primary driver of signal swing, power dissipation, and architecture selection not only in filter circuits but in analog circuit design in general

Review from last lecture

Statistical Characterization of Noise



$v_n(t)$



$$v_{RMS} = \sqrt{\left(\int_0^{\infty} S df \right)} \quad \overset{?}{\Longrightarrow} \quad v_{RMS} = \lim_{T \to \infty} \int_{t_1}^{t_1+T} v^2(t) dt$$

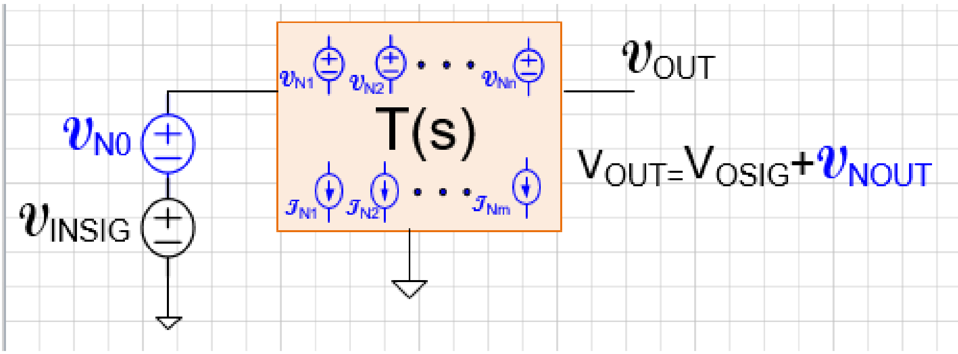
Parseval's Theorem

$$\sqrt{\int_{f=0}^{\infty} Sdf} = \lim_{T \to \infty} \int_{t_1}^{t_1+T} v^2(t) dt$$

Review from last lecture

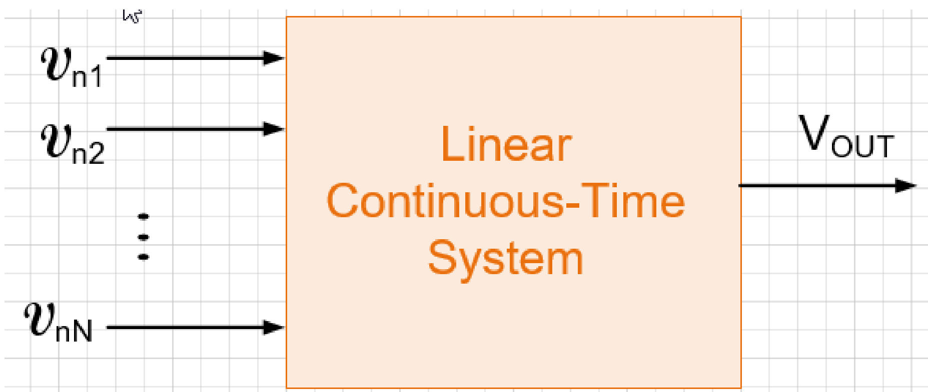
Analysis of Noise in Filter Circuits

Consider a filter circuit with N noise voltage sources (can be easily modified to include both noise voltage and current sources)



N=n+1

The noise sources can be represented by the block diagram shown below



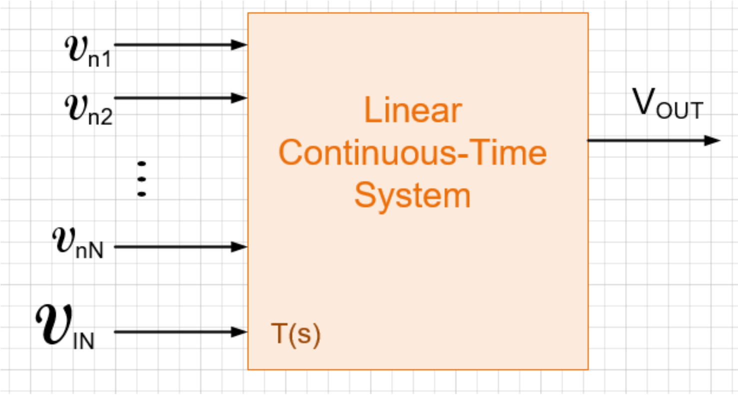
Assume $T_k(s)$ is the transfer function from the kth source to the output

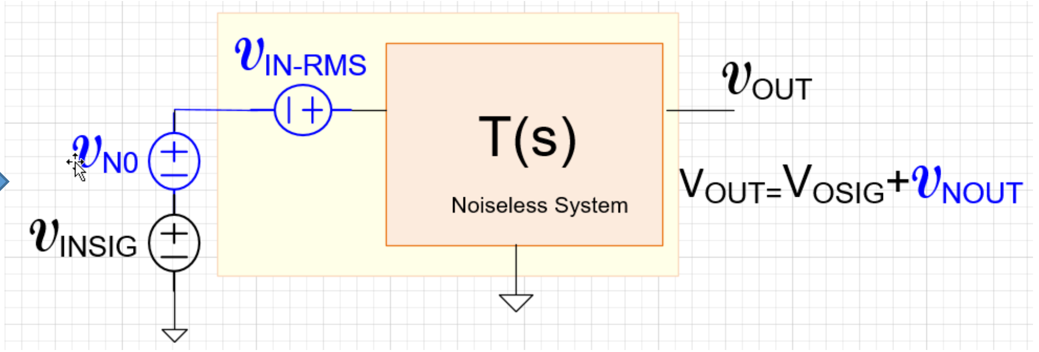
By superposition

$$V_{OUT}(s) = \sum_{i=1}^{N} T_i(s) V_i(s)$$

Review from last lecture

Input-Referred Noise in Filter Circuits





$$v_{OUT\_RMS}=\sqrt{\int_{f=0}^{\infty}S_{OUT}df}=\sqrt{\int_{f=0}^{\infty}\sum_{i=1}^{N}S_i\bullet\left|T_i\left(j\omega\right)\right|^2df}$$

Let T(s) be the transfer function from the input to the output. (usually T(s) will be distinct from each of the noise transfer functions).

The input-referred noise spectral density is given by the expression

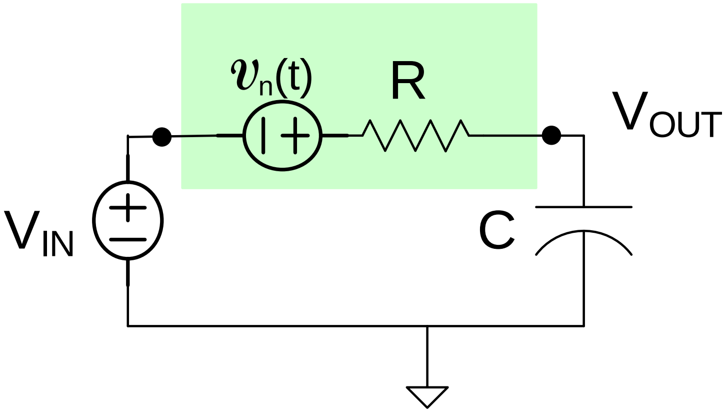
$$S_{IN}=\frac{S_{OUT}}{\left|T\left(j\omega\right)^2\right|}$$

The input-referred RMS voltage is thus given by

$$v_{IN\_RMS}=\sqrt{\int_{f=0}^{\infty}\frac{S_{OUT}}{\left|T\left(j\omega\right)\right|^2}df}=\sqrt{\int_{f=0}^{\infty}\sum_{i=1}^{N}S_i\bullet\frac{\left|T_i\left(j\omega\right)\right|^2}{\left|T\left(j\omega\right)\right|^2}df}$$

Review from last lecture

Example: First-Order RC Network



$$v_{n_{RMS}} = \sqrt{\int_{f=0}^{\infty} S_{V_{OUT}}\, df} = \sqrt{\int_{f=0}^{\infty} \frac{4kTR}{1+\omega^2 R^2 C^2}\, df}$$

From a standard change of variable with a trig identity, it follows that

$$v_{n_{RMS}} = \sqrt{\int_{f=0}^{\infty} S_{V_{OUT}}\, df} = \sqrt{\frac{kT}{C}}$$

- Note the continuous-time noise voltage has an RMS value that is independent of R
- The noise contributed by the resistor is dependent only upon the capacitor value C
- This is often referred to at kT/C noise and it can be decreased at a given T only by increasing C

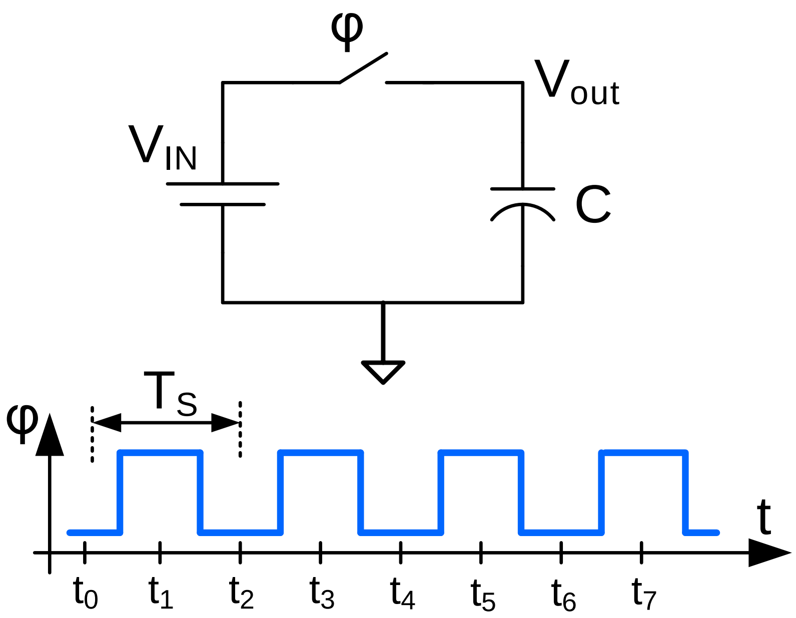
Review from last lecture

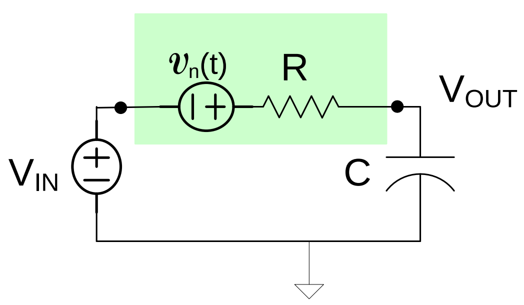
Theorem If $\mathcal{V}(t)$ is a continuous-time zero-mean noise source and <$\mathcal{V}(kT)$> is a sampled version of $\mathcal{V}(t)$ sampled at times T, 2T, …. then the RMS value of the continuous-time waveform is the same as that of the sampled version of the waveform. This can be expressed as $\mathcal{V}_{RMS} = \hat{\mathcal{V}}_{RMS}$

Theorem If $\mathcal{V}(t)$ is a continuous-time zero-mean noise signal and <$\mathcal{V}(kT)$> is a sampled version of $\mathcal{V}(t)$ sampled at times T, 2T, …. then the standard deviation of the random variable $\mathcal{V}(kT)$, denoted as $\sigma_{\hat{V}}$ satisfies the expression $\sigma_{\hat{V}} = \mathcal{V}_{RMS} = \hat{\mathcal{V}}_{RMS}$

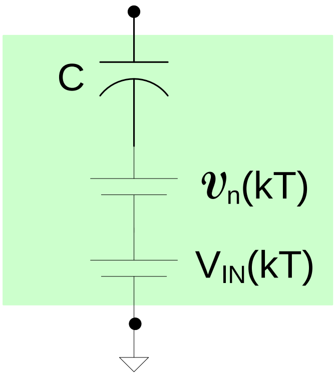
Review from last lecture

Example: Switched Capacitor Sampler





Track mode

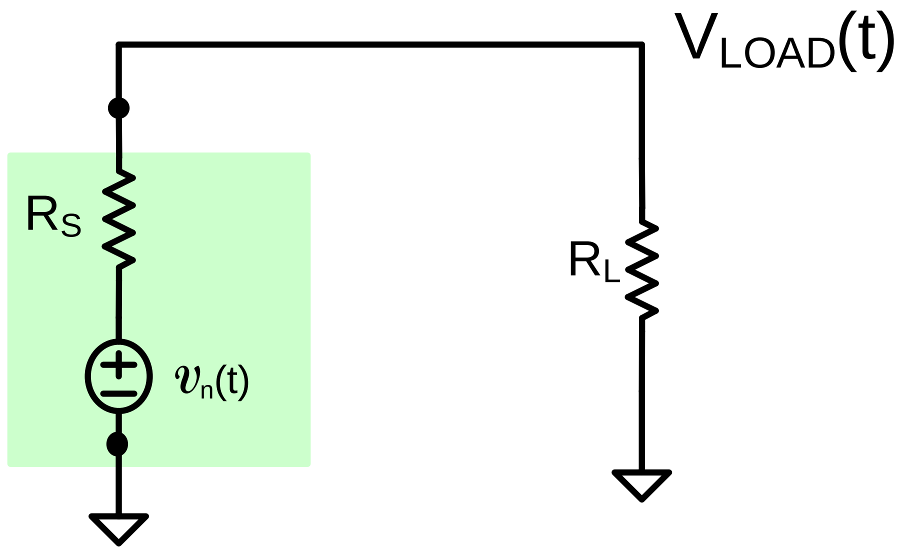


Hold mode

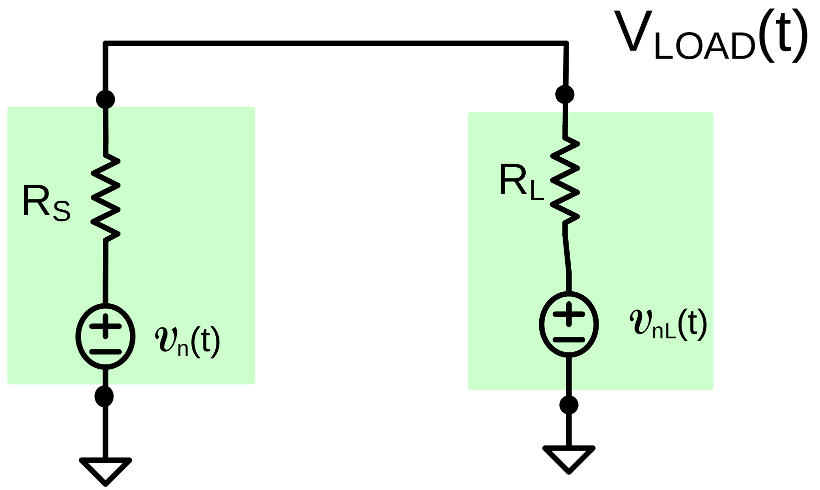
$$v_{n_{RMS}} = \sqrt{\frac{kT}{C}}$$

Review from last lecture

What is the RMS value of the output noise voltage due to the noise on R_S?



What is the RMS value of the output noise voltage due to the noise on R_L and R_S?



Digital Filters

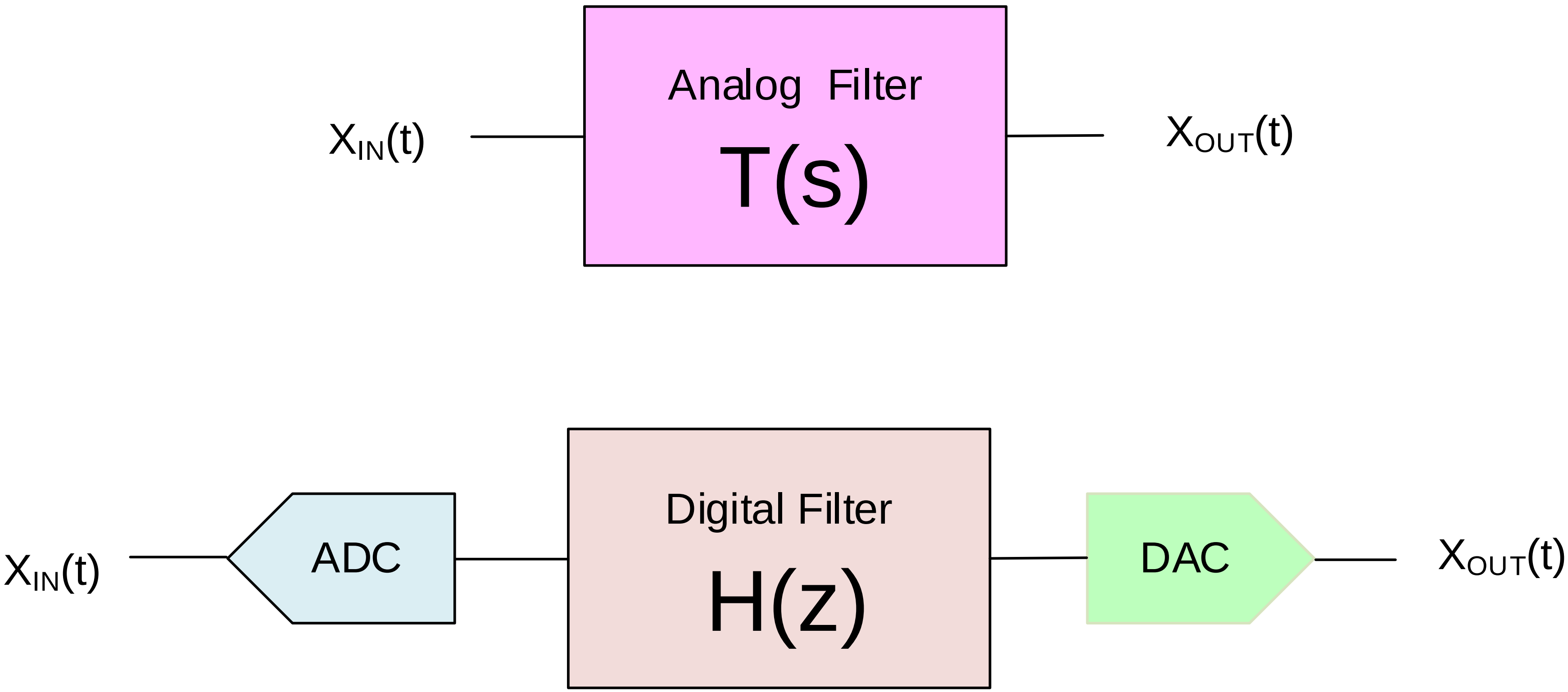
Limitations of Analog Filters

- Transfer functions sensitive to component and process variations
- Distortion inherent due to nonlinearities in components (particularly amplifiers)
- Power dissipation can be large
- Area gets large, often unacceptably so for very low frequency poles and even of concern for audio-frequency poles
- Programmability introduces considerable complexity (with existing approaches)
- Making minor changes in filter requirements often necessitates a major redesign effort

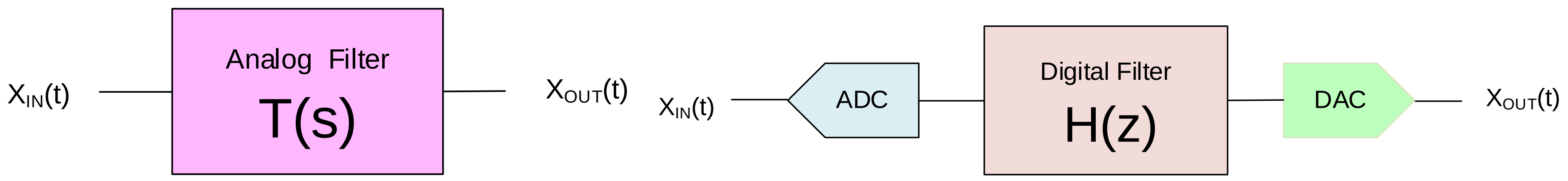
Advantages of Analog Filters

Significant advantages but since comparative, will discuss later

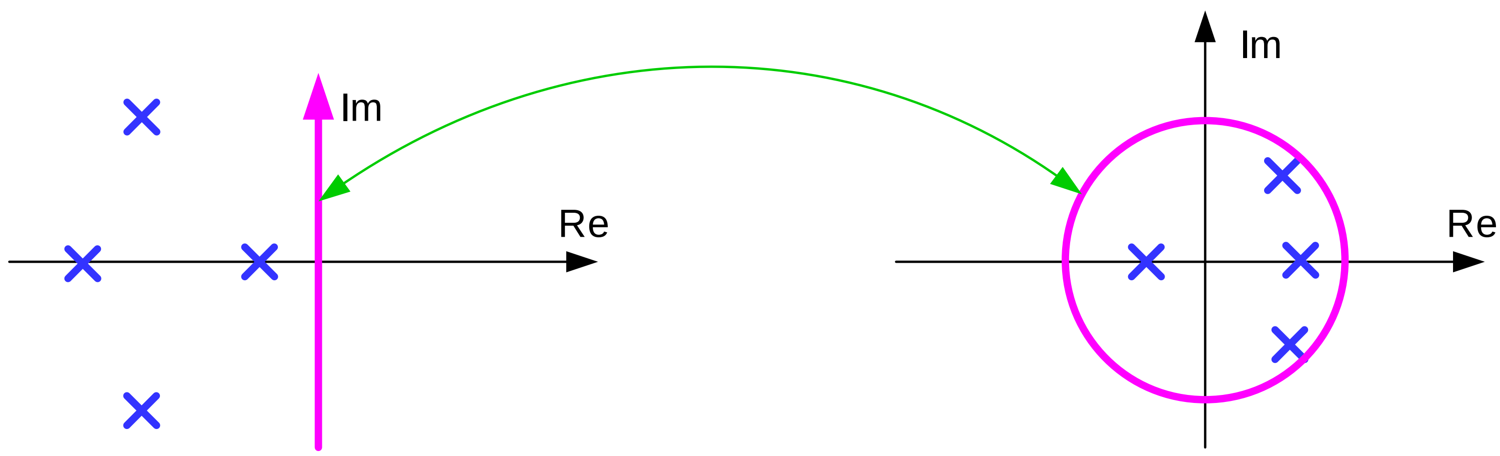
Analog vs Digital Filter



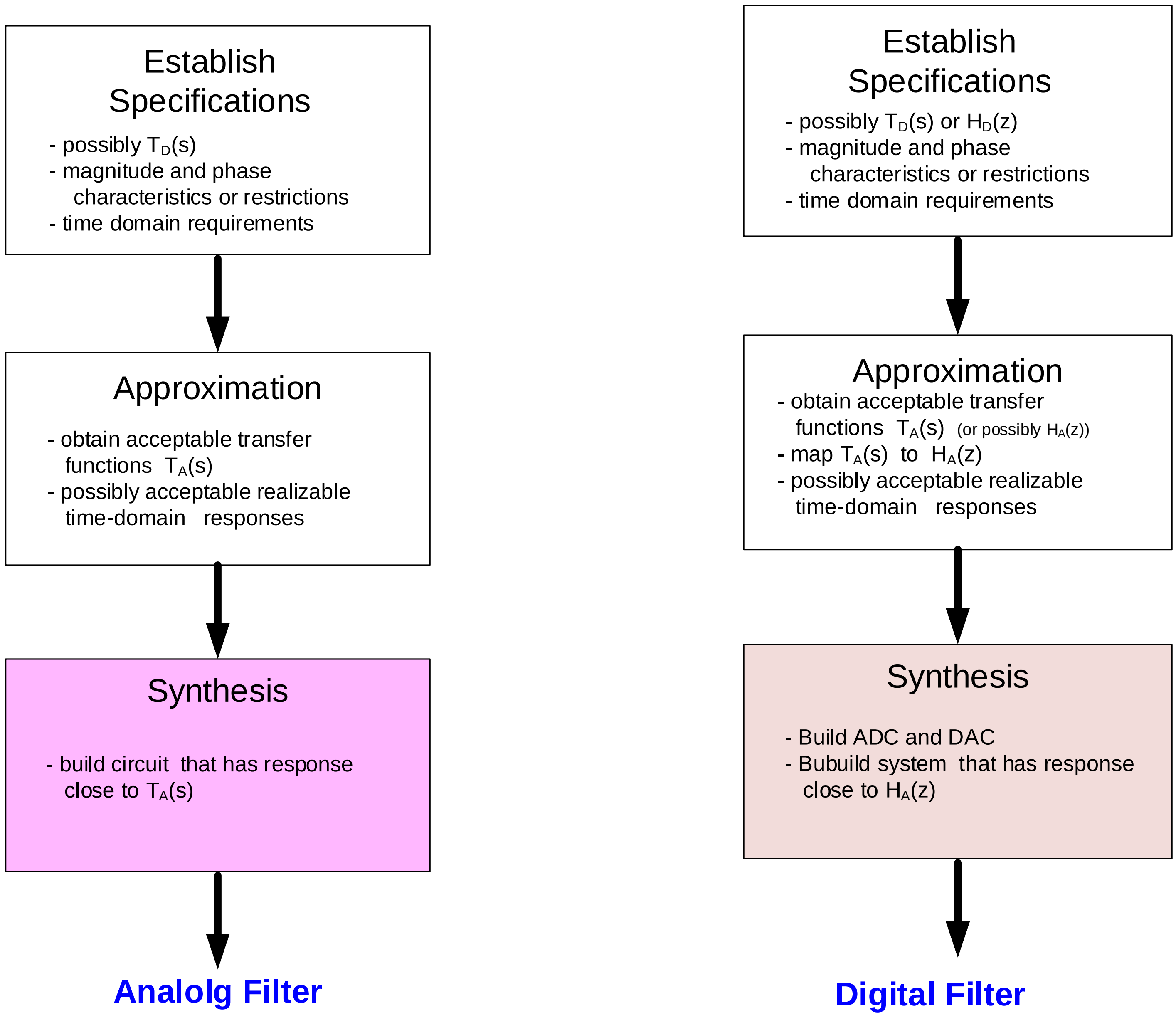
Analog vs Digital Filter



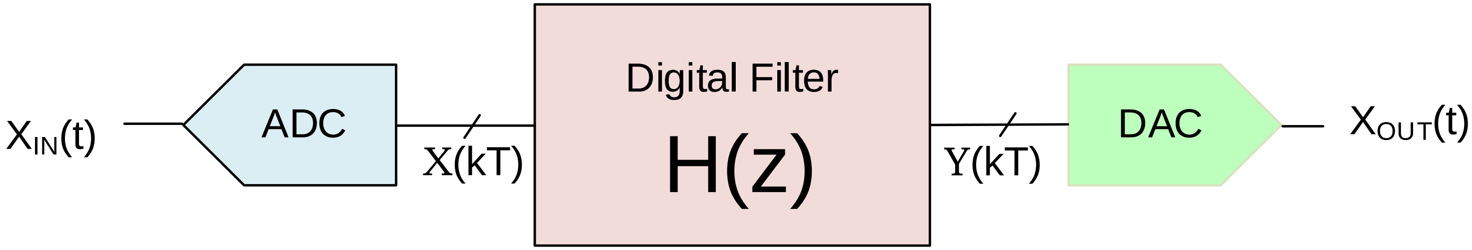
- Imaginary axis for analog filters corresponds to unit circle for digital filters
- Several standard mappings used to map between s-domain and z-domain
- Some actually map imaginary axis to unit circle



Filter Design Processes



Digital Filter Properties



$$H(z) = \frac{\sum_{i=0}^{m} \alpha_i z^{-i}}{\sum_{i=0}^{n} \beta_i z^{-i}}$$

for $\beta_0=1$ and $a_i=\alpha_i$, $b_i=\beta_i$

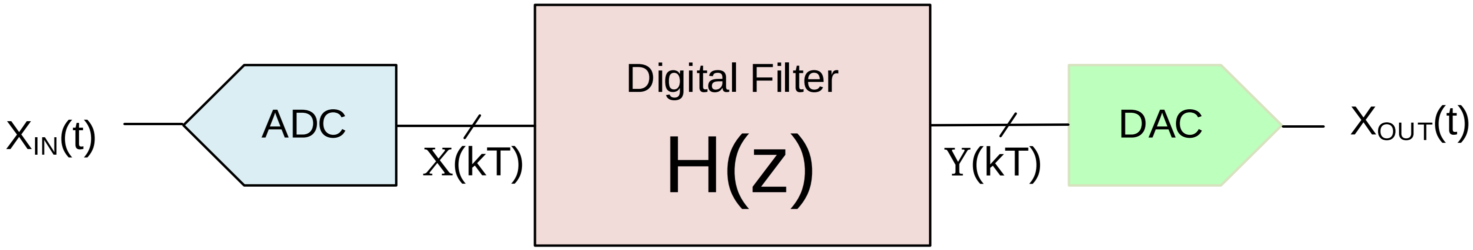
$$y(nT) = \sum_{i=0}^{m} a_i x(nT - iT) + \sum_{i=1}^{n} b_i y(nT - iT)$$

If n=1 termed a Moving Average (MA) - only zeros present

If m=1 termed an Auto Regressive (AR) - only poles present

If mn $\neq 1$ termed Autoregressive Moving Average (ARMA) – poles and zeros present

Digital Filter Properties



$$H(z) = \frac{\sum_{i=0}^{m} \alpha_i z^{-i}}{\sum_{i=0}^{n} \beta_i z^{-i}}$$

If n=1 termed a Moving Average (MA) - only zeros present

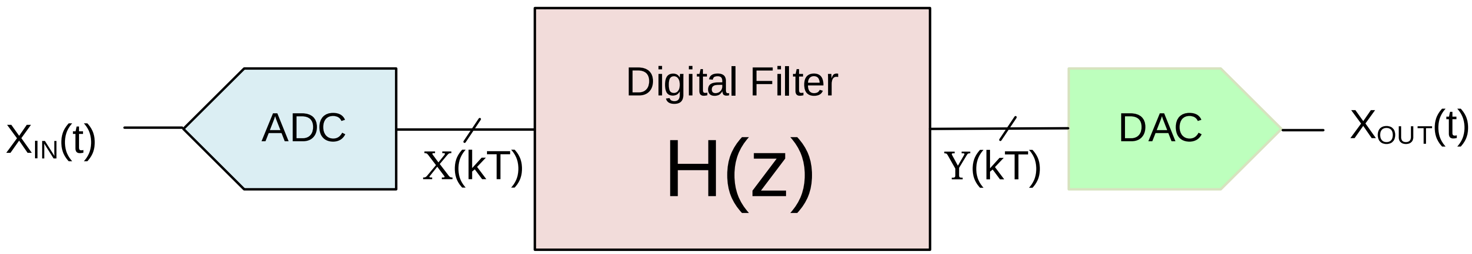
MA filters have a finite impulse response so are often termed FIR filters

If m=1 or mn $\neq 1$ the filter will have one or more poles (and possibly many zeros but not necessarily any)

AR and ARMA filters have infinite impulse response so are often termed IIR filters

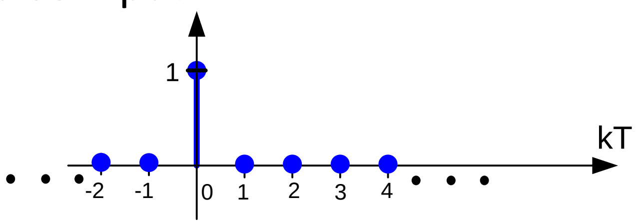
Digital Filter Properties

Moving Average Filters



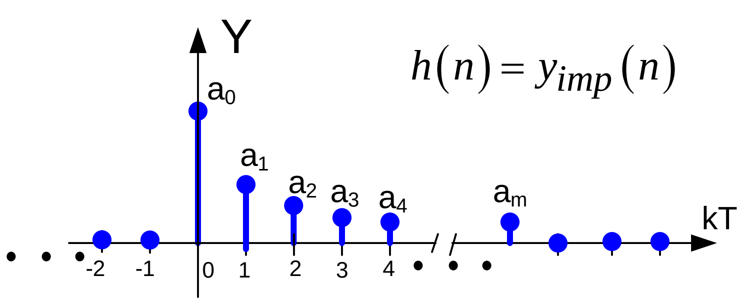
$$H(z) = \sum_{i=0}^{m} a_i z^{-i} \quad \Rightarrow \quad y(nT) = \sum_{i=0}^{m} a_i x(nT - iT)$$

Impulse Input



$$x_{imp}(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Impulse Response



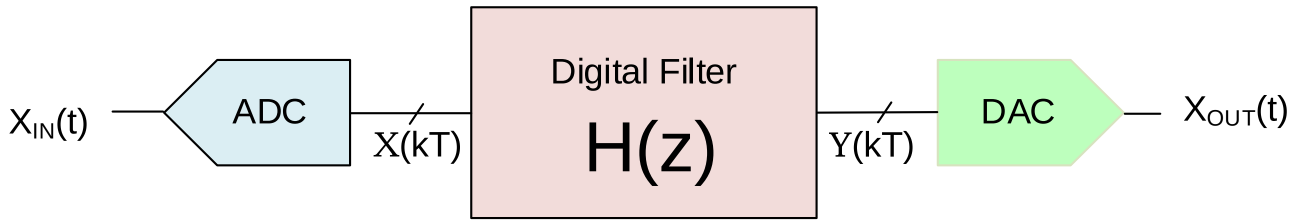
$$h(n) = y_{imp}(n)$$

It can be shown that

$$h(n) = \begin{cases} a_n & 0 \le n \le m \\ 0 & n < 0 \ \ and \ \ n > m \end{cases}$$

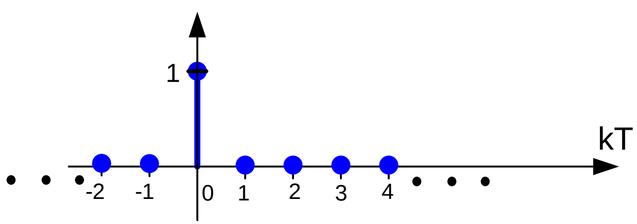
Digital Filter Properties

Moving Average Filters

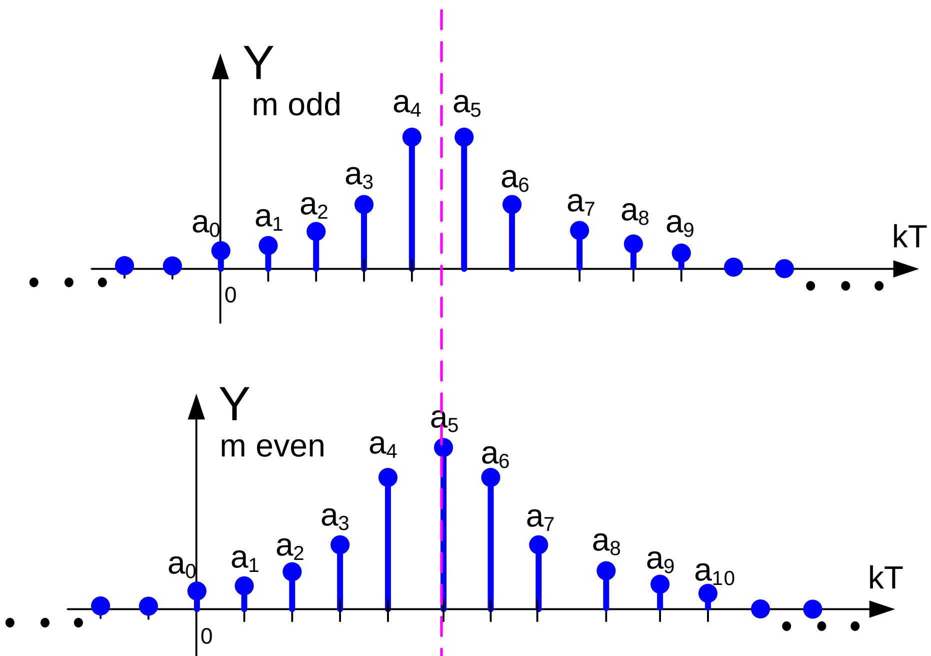


$$y(nT) = \sum_{i=0}^{m} a_i x(nT - iT)$$

Impulse Input



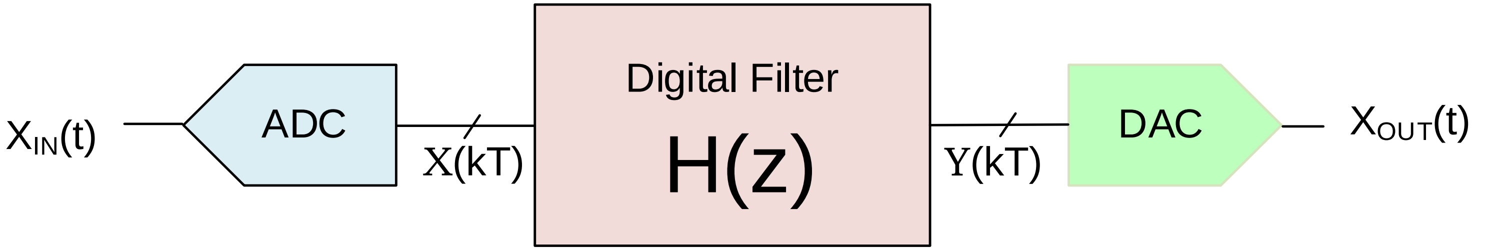
Impulse Response



Impulse response often symmetric around “m/2”

Digital Filter Properties

Moving Average Filters



$$\mathrm{H}(z) = \sum_{i=0}^{m} a_i z^{-i} \quad \Rightarrow \quad y(nT) = \sum_{i=0}^{m} a_i x(nT - iT)$$

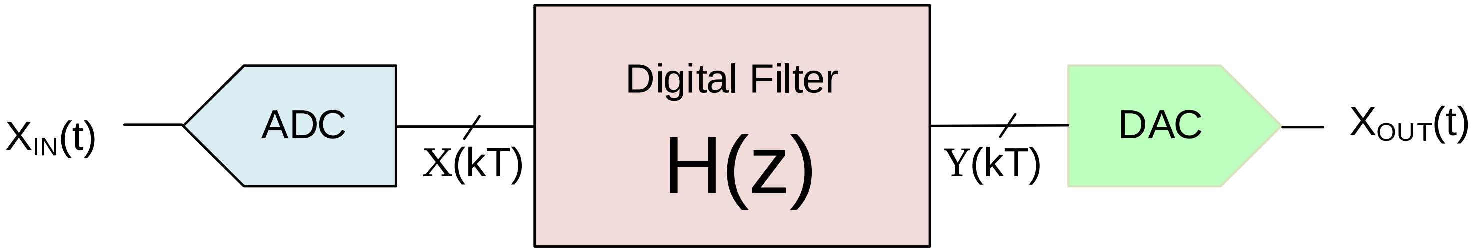
It can be shown that

$$y(n) = x(n) \otimes h(n) \underset{def}{=} \sum_{k=0}^{m} h(k)\, x(n-k)$$

FIR filters are sometimes termed convolutional filters

Digital Filter Properties

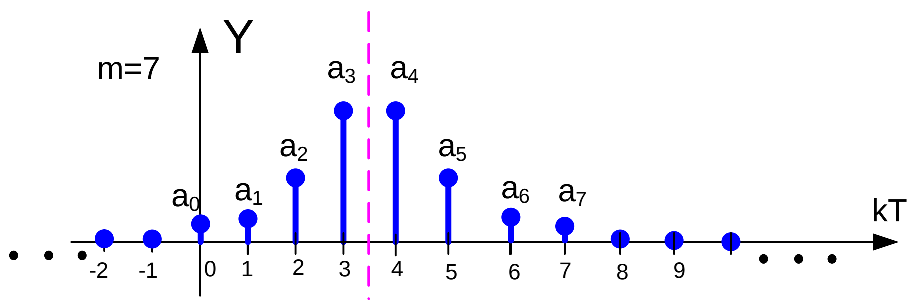
Moving Average Filters



$$H(z) = \sum_{i=0}^{m} a_i z^{-i} \quad \Rightarrow \quad y(nT) = \sum_{i=0}^{m} a_i x(nT - iT)$$

FIR Filters can be easily designed to have linear phase

Example: m=7

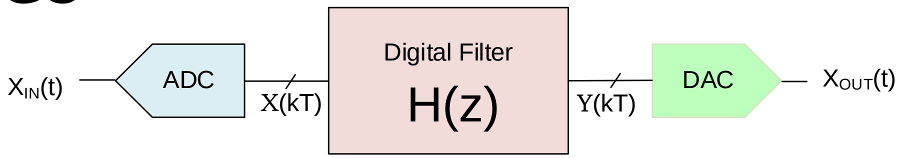


$$H(z) = \sum_{i=0}^{m} \alpha_i z^{-i} = a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + a_4 z^{-4} + a_5 z^{-5} + a_6 z^{-6} + a_7 z^{-7}$$

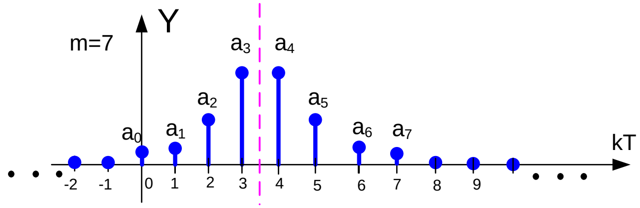
$$H(z) = a_0\left[1 + z^{-7}\right] + a_1\left[z^{-1} + z^{-6}\right] + a_2\left[z^{-2} + z^{-5}\right] + a_3\left[z^{-3} + z^{-4}\right]$$

Digital Filter Properties

Moving Average Filters



FIR Filters can be easily designed to have linear phase



Example: m=7

$$\mathrm{H}(z) = a_0\left[1 + z^{-7}\right] + a_1\left[z^{-1} + z^{-6}\right] + a_2\left[z^{-2} + z^{-5}\right] + a_3\left[z^{-3} + z^{-4}\right]$$

$$\mathrm{H}(z)\Big|_{z=e^{j\omega}} = a_0\left[1 + e^{-7j\omega}\right] + a_1\left[e^{-j\omega} + e^{-6j\omega}\right] + a_2\left[e^{-2j\omega} + e^{-5j\omega}\right] + a_3\left[e^{-3j\omega} + e^{-4j\omega}\right]$$

This can be rewritten as

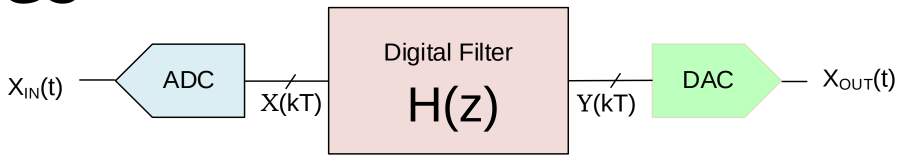
$$\mathrm{H}\left(e^{j\omega}\right) = a_0 e^{-\frac{7}{2}j\omega}\left[e^{\frac{7}{2}j\omega} + e^{-\frac{7}{2}j\omega}\right] + a_1 e^{-\frac{7}{2}j\omega}\left[e^{\frac{5}{2}j\omega} + e^{-\frac{5}{2}j\omega}\right] + a_2 e^{-\frac{7}{2}j\omega}\left[e^{\frac{3}{2}j\omega} + e^{-\frac{3}{2}j\omega}\right] + a_3 e^{-\frac{7}{2}j\omega}\left[e^{\frac{1}{2}j\omega} + e^{-\frac{1}{2}j\omega}\right]$$

By Euler's Formula $\cos x = \dfrac{e^{jx} + e^{-jx}}{2}$

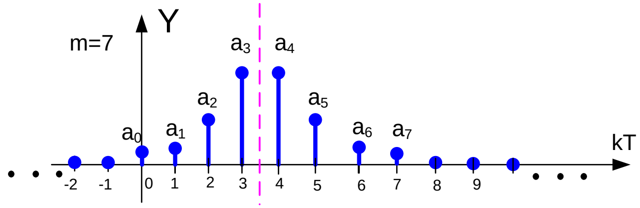
$$\mathrm{H}\left(e^{j\omega}\right) = a_0 e^{-\frac{7}{2}j\omega}\left[2\cos\left(\frac{7}{2}\omega\right)\right] + a_1 e^{-\frac{7}{2}j\omega}\left[2\cos\left(\frac{5}{2}\omega\right)\right] + a_2 e^{-\frac{7}{2}j\omega}\left[2\cos\left(\frac{3}{2}\omega\right)\right] + a_3 e^{-\frac{7}{2}j\omega}\left[2\cos\left(\frac{1}{2}\omega\right)\right]$$

Digital Filter Properties

Moving Average Filters



FIR Filters can be easily designed to have linear phase



Example: m=7

$$\mathrm{H}\left(\mathrm{e}^{j\omega}\right)=a_0 e^{-\frac{7}{2}j\omega}\left[2\cos\left(\frac{7}{2}\omega\right)\right]+a_1 e^{-\frac{7}{2}j\omega}\left[2\cos\left(\frac{5}{2}\omega\right)\right]+a_2 e^{-\frac{7}{2}j\omega}\left[2\cos\left(\frac{3}{2}\omega\right)\right]+a_3 e^{-\frac{7}{2}j\omega}\left[2\cos\left(\frac{1}{2}\omega\right)\right]$$

Regrouping, we obtain

$$\mathrm{H}\left(\mathrm{e}^{j\omega}\right)=e^{-\frac{7}{2}j\omega}2\left(a_0\cos\left(\frac{7}{2}\omega\right)+a_1\cos\left(\frac{5}{2}\omega\right)+a_2\cos\left(\frac{3}{2}\omega\right)+a_3\cos\left(\frac{1}{2}\omega\right)\right)$$

It thus follows that

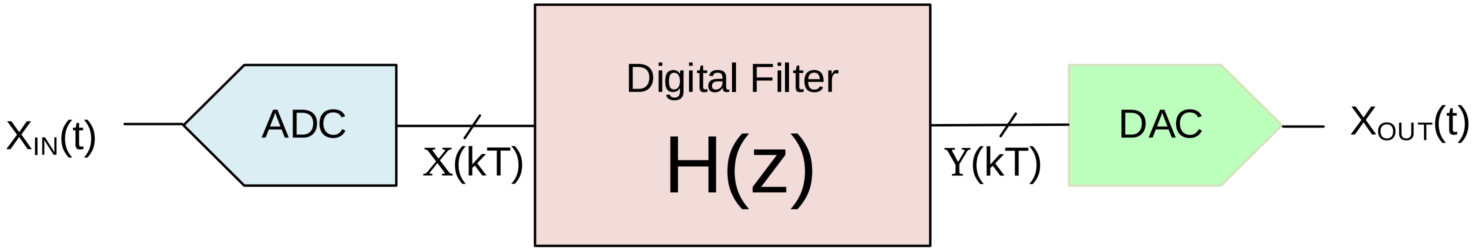
$$\left|\mathrm{H}\left(\mathrm{e}^{j\omega}\right)\right|=2\left(a_0\cos\left(\frac{7}{2}\omega\right)+a_1\cos\left(\frac{5}{2}\omega\right)+a_2\cos\left(\frac{3}{2}\omega\right)+a_3\cos\left(\frac{1}{2}\omega\right)\right)$$

$$\angle\mathrm{H}\left(\mathrm{e}^{j\omega}\right)=-\frac{7}{2}\omega$$

Thus H(z) is linear phase !

This property holds for any symmetric impulse response of a FIR filter of any order

Digital Filter Properties



It is easy to design linear phase digital filters

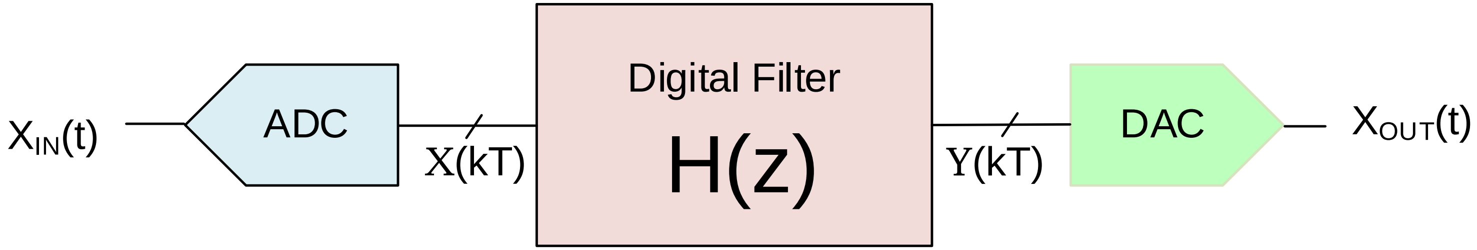
$$y(nT) = \sum_{i=0}^{m} a_i x(nT - iT)$$

Theorem: Any FIR filter is linear phase if the impulse response is symmetric or antisymmetric

Type	impulse response	
I	symmetric	length is odd
II	symmetric	length is even
III	anti-symmetric	length is odd
IV	anti-symmetric	length is even

Table from Robert Novak book

Digital Filter Properties



Theorem: Any FIR filter is linear phase if the impulse response is symmetric or antisymmetric

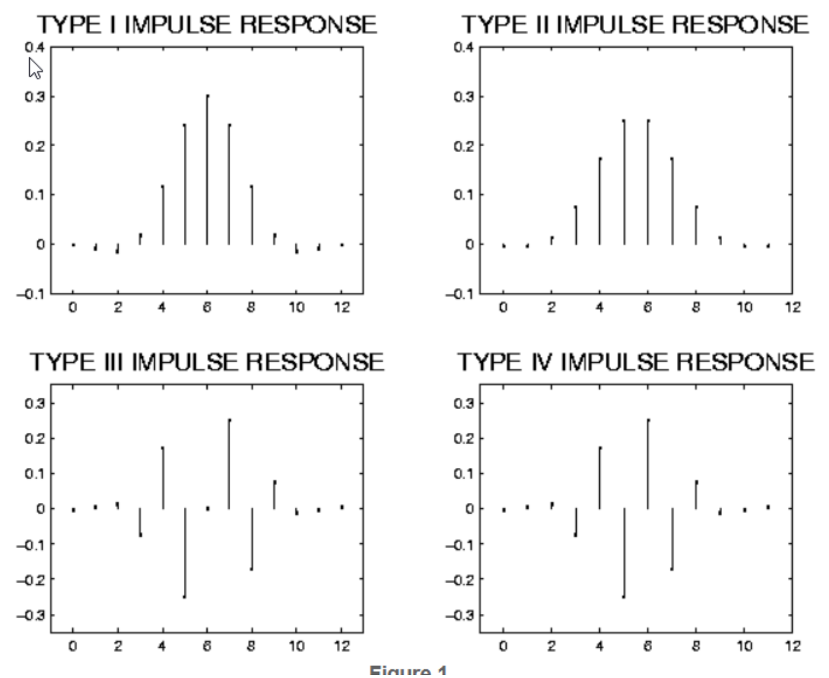
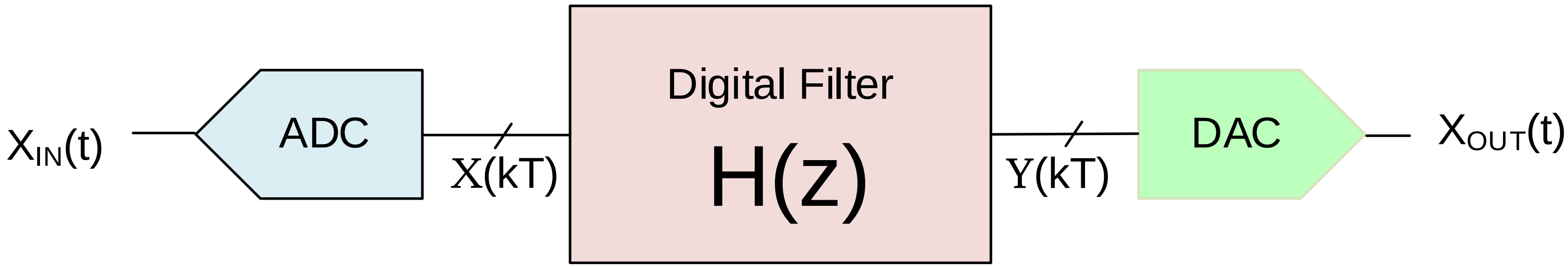


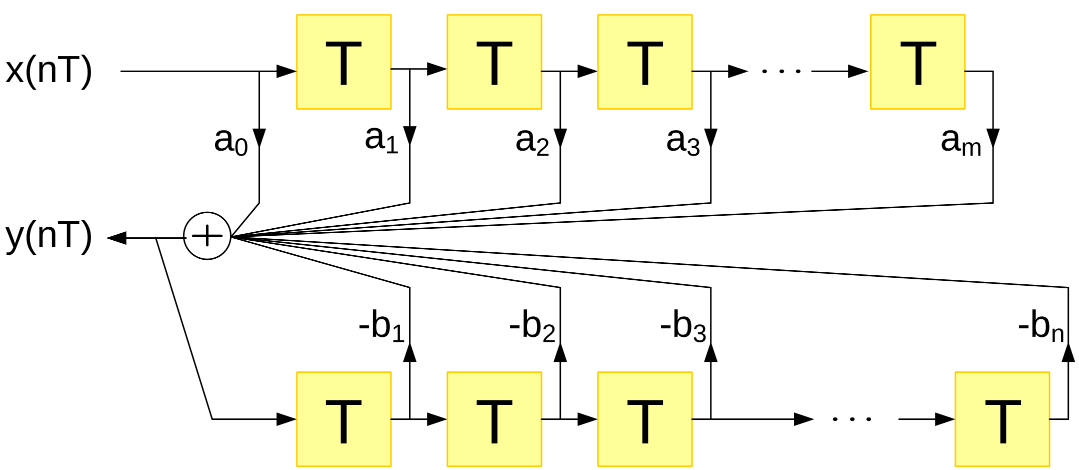
Table from Robert Novak book

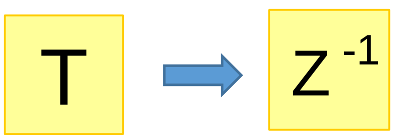
Digital Filter Properties



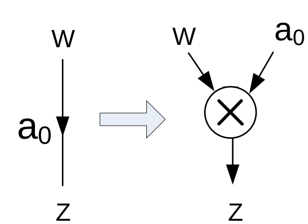
$$y(nT) = \sum_{i=0}^{m} a_i x(nT - iT) + \sum_{i=1}^{n} b_i y(nT - iT)$$

An Implementation of a Digital Filter



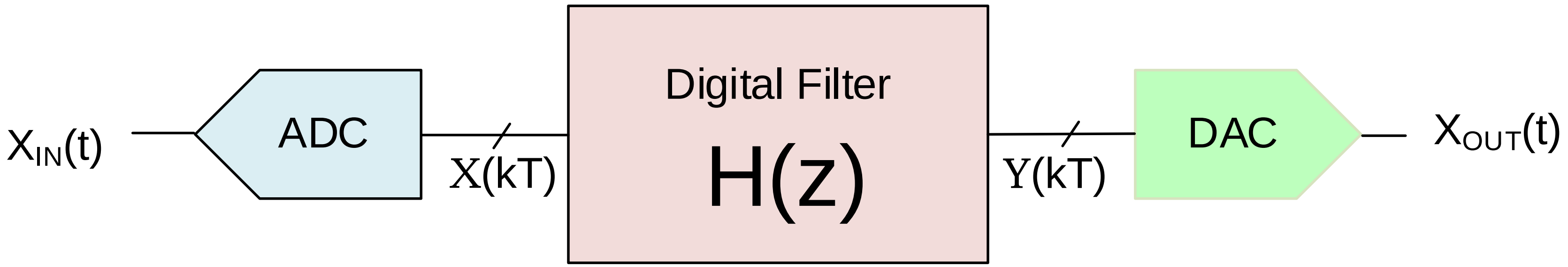


Delay Element



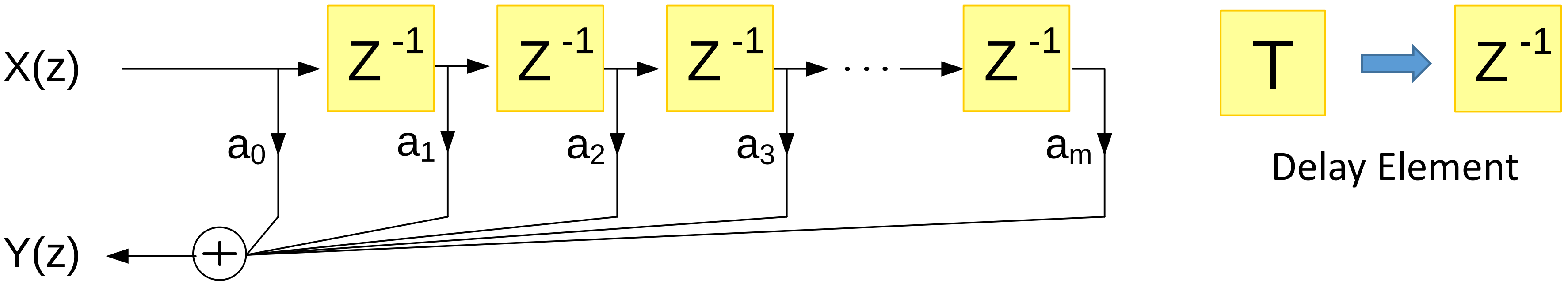
Multiply Element

Digital Filter Properties



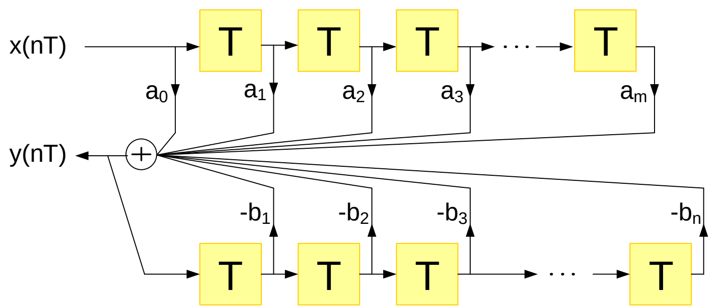
An Implementation of a FIR Digital Filter

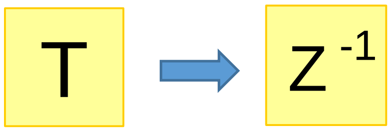
$$y(nT) = \sum_{i=0}^{m} a_i x(nT - iT)$$



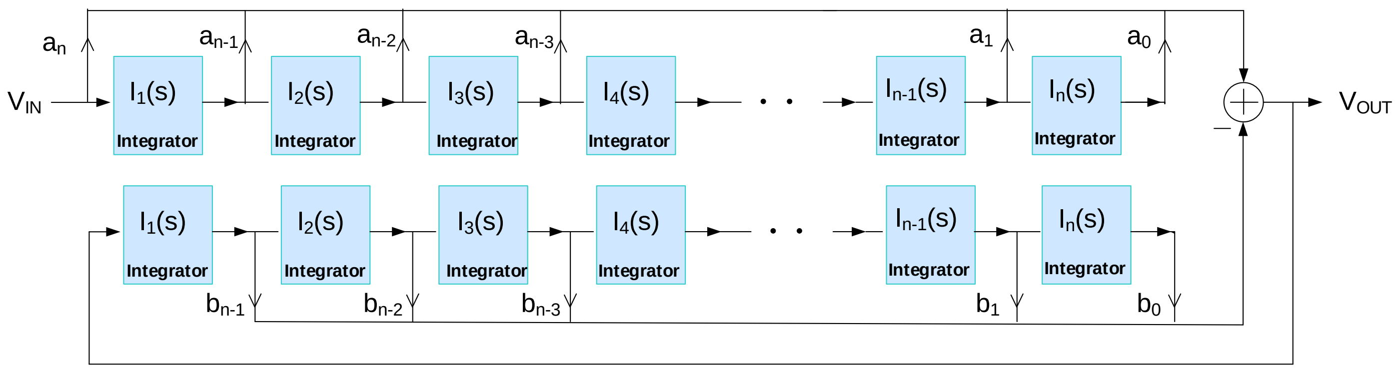
- Delay operations or delay filters are easily implemented with digital filters
- Delay for each delay element is one clock period

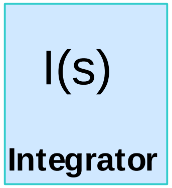
An Implementation of a Digital Filter





An Implementation of an Analog Filter





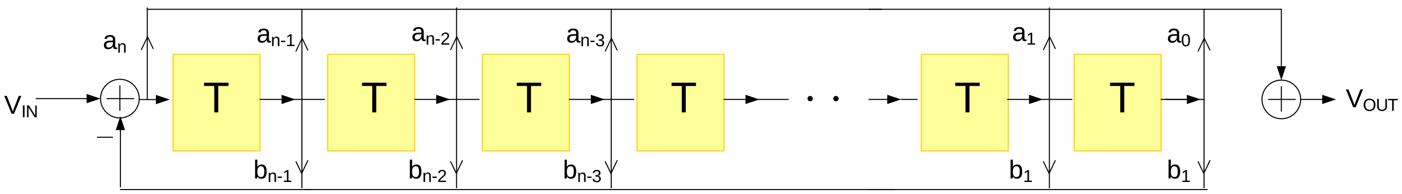
- Can be viewed as analogous implementations
- Neither particularly practical
- Many other architectures for both analog and digital filters
- Approximately double the number of integrators or delay elements needed

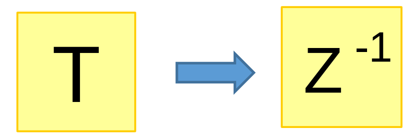
An Implementation of a Digital Filter

$$H(z)=\frac{\sum_{i=0}^{m}\alpha_i z^{-i}}{\sum_{i=0}^{n}\beta_i z^{-i}}$$

for $\beta_0=1$ and $a_i=\alpha_i$, $b_i=\beta_i$

$$y(nT)=\sum_{i=0}^{m}a_i x(nT-iT)+\sum_{i=1}^{n}b_i y(nT-iT)$$



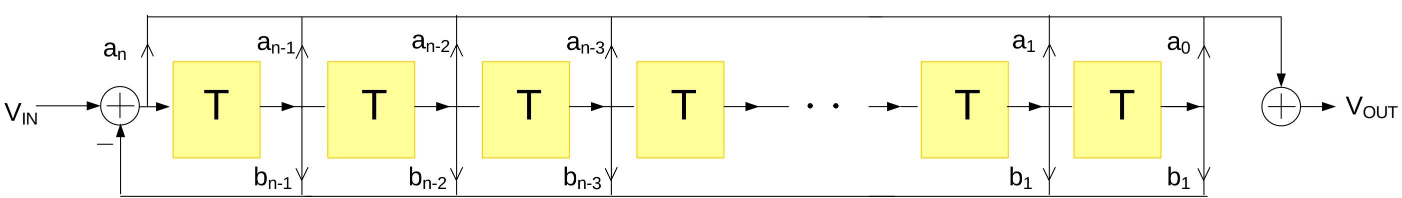


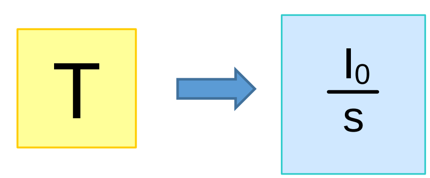
Delay Element

An Implementation of an Analog Filter

$$T(s)=\frac{\sum_{k=0}^{n}a_{n-k}I_0^k s^{n-k}}{s^n+\sum_{k=1}^{n}b_{n-k}I_0^k s^{n-k}}$$

$$\frac{d^n v_{OUT}}{dt^n}=\sum_{k=0}^{m}a_k\frac{d^k v_{IN}}{dt^k}-\sum_{k=0}^{n-1}b_k\frac{d^k v_{OUT}}{dt^k}$$

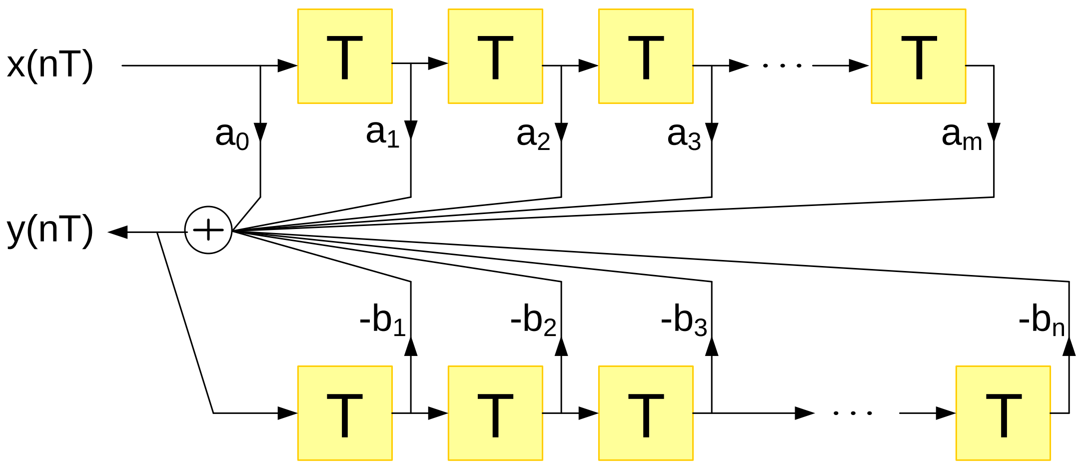


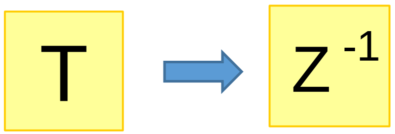


Integrator Element

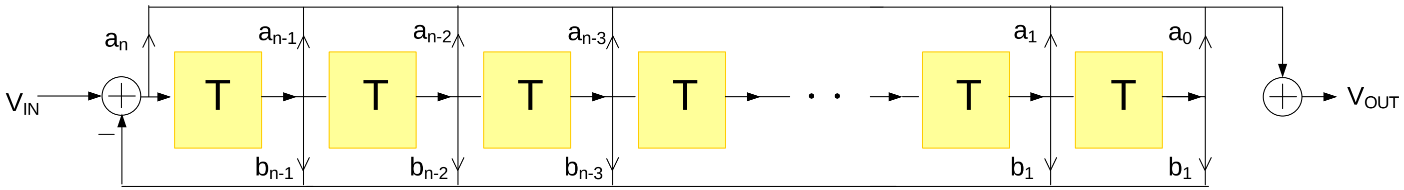
Termed Direct Synthesis or Analog Computer Approach

Alternate Implementations of an Digital Filter



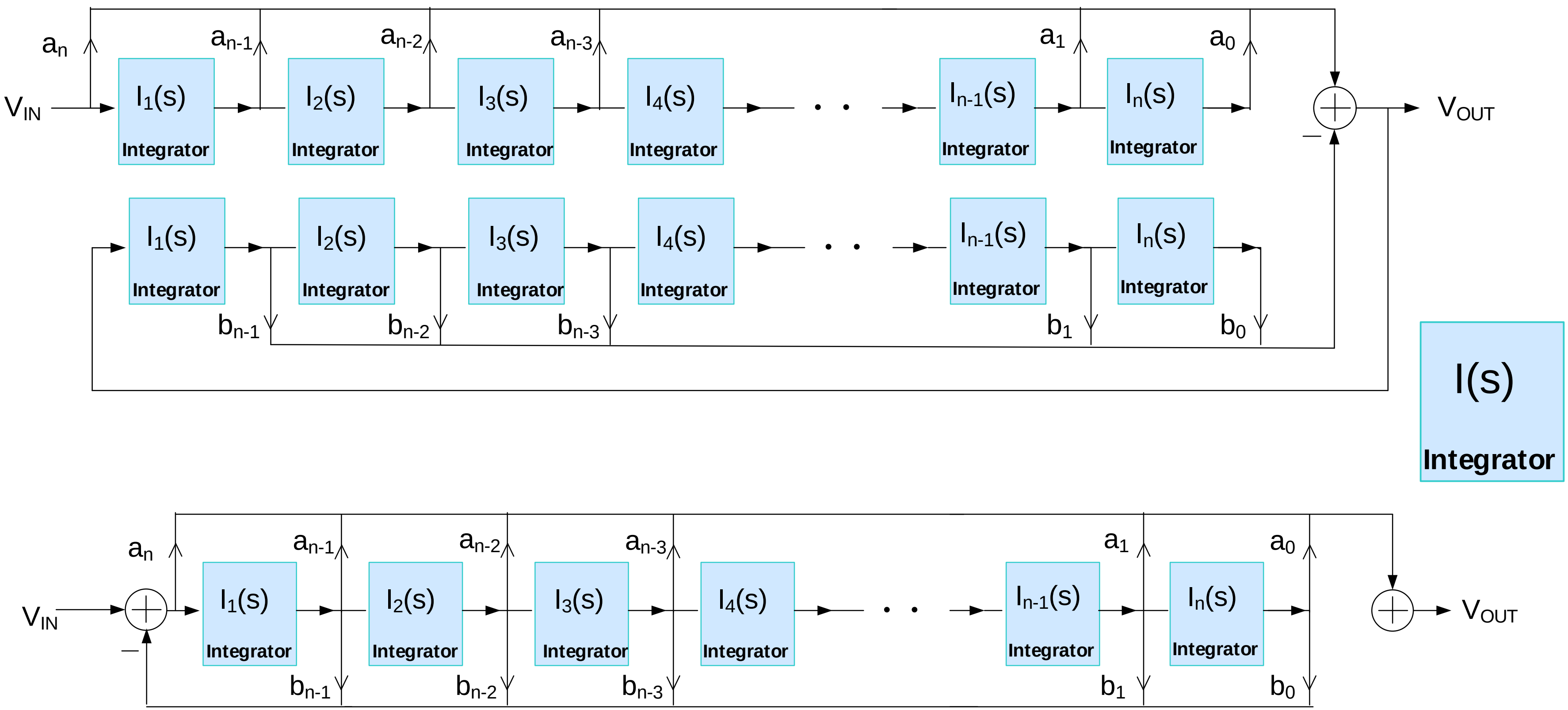


Delay Element



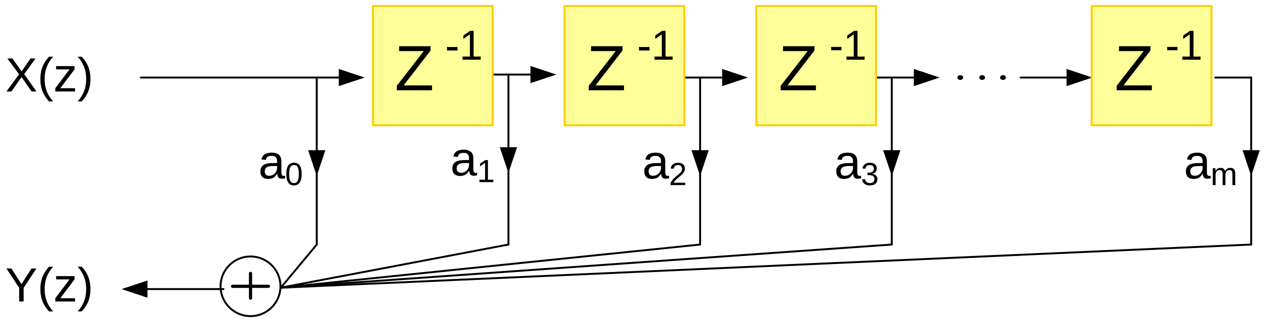
- Reduced number of delay elements by factor of 2
- Still not particularly practical
- Similar architectural change can be made for analog filter (next slide)

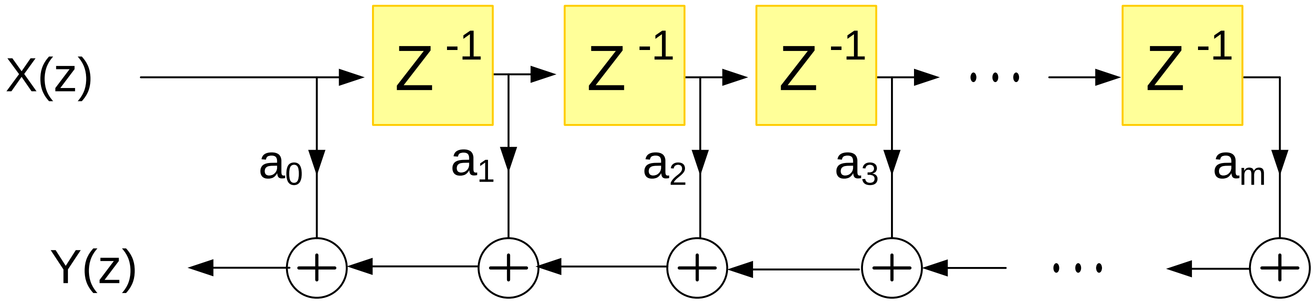
Alternate Implementations of an Analog Filter



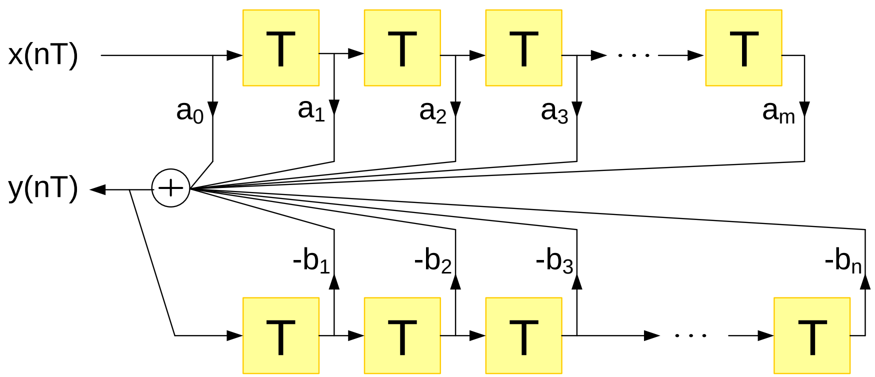
- Reduced number of integrators by factor of 2
- Still not particularly practical
- Similar architectural change for digital filter (previous slide)

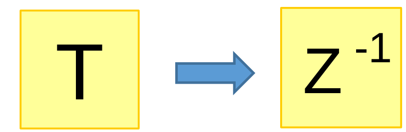
Alternate Implementations of an FIR Digital Filter

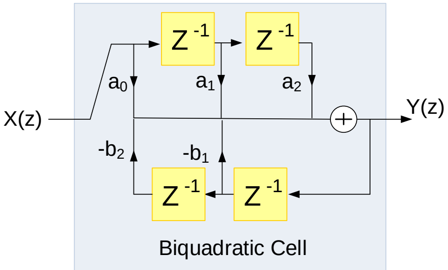


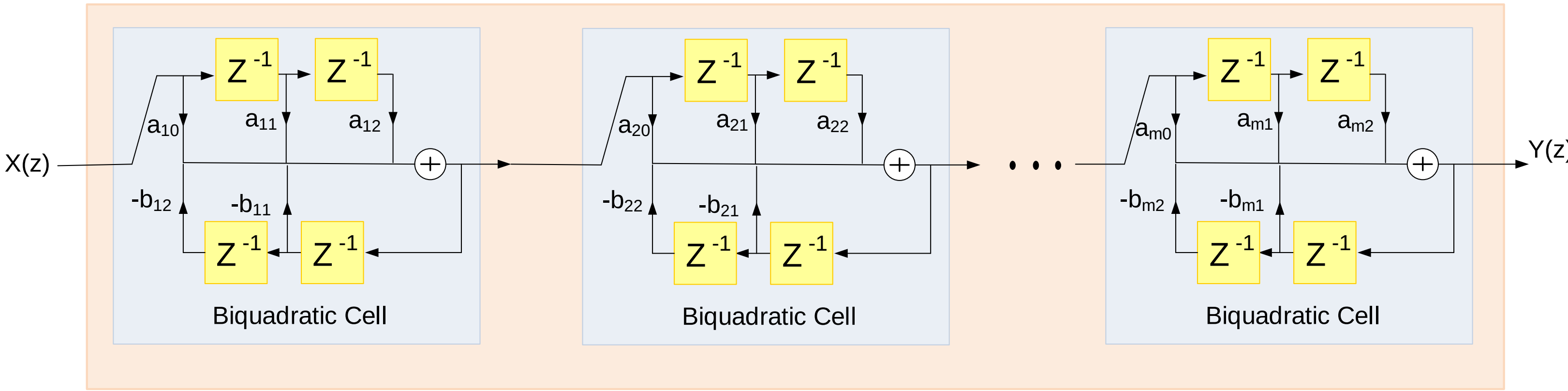


Alternate Implementations of IIR Digital Filter



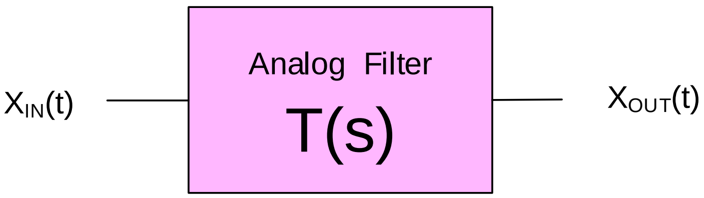


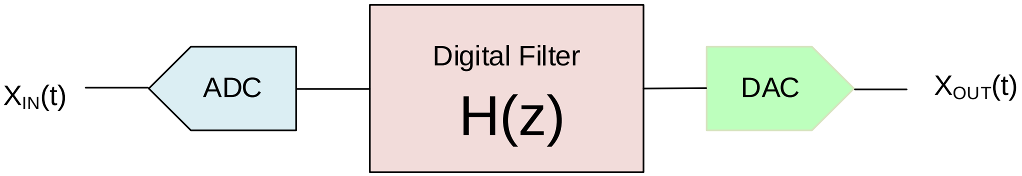




Excessive delay elements but not of as much concern as excessive Integrators

Does Digital Filter Overcome Limitations





A - Transfer functions sensitive to component and process variations

D - Transfer function part of H(z) not sensitive to process variations
- Transfer function sensitive to coefficient quantization
- ADC and DAC minimally sensitive to process variations but highly sensitive to missmatch

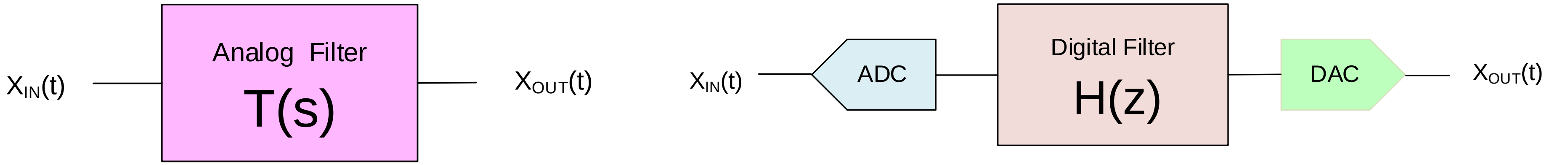
A - Distortion inherent due to nonlinearities in components (particularly amplifiers)

D - Transfer function part of H(z) not sensitive nonlinearity of components
- ADC and DAC sensitive to nonlinearity of components

A - Power dissipation can be large

D - Power dissipation can be large due to a large number of arithmetic operations during each clock cycle
- ADC and DAC dissipate considerable energy for high resolution or high speed

Does Digital Filter Overcome Limitations



A - Area gets large, often unacceptably so for very low frequency poles and even of concern for audio-frequency poles

D - Area for DSP in Digital Filter can be large
- ADC and DAC can become large if high resolution is required
- No area penalty for low frequency operation of digital system

A - Programmability introduces considerable complexity (with existing approaches)

D - Programmability of filter characteristics is very efficient with digital filter approach

A - Making minor changes in filter requirements often necessitates a major redesign effort

D - Making minor or even major changes in filter requirements requires minimal effort with digital filter approach

Stay Safe and Stay Healthy !

End of Lecture 37